SET – ONE

MODAL QUESTIONS

OF

P.G. sem – III, PAPER – 308

MATHEMATICS

PREPARED

BY

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For, Kolhan university, Chaibasa

Time : 3 hours

Full Marks : 70

Candidates are required to give their answers in their own words as far as practible.

The figures in the margin indicate full marks.

Answer from both the parts A and B.

Part A contains multi-choice questions.

PART – A.

1. Answer all questions: $1 \times 10 = 10$

(a) $||x+y|| \leq ||x|| + ||y||$ is called

(i) Absolute homogenity (ii) Sub additivity
(iii) Linearity (iv) None of these

(b) For $1 \leq p < \infty$, then L_p space is

(i) Continuous (ii) Convergent
(iii) Complete (iv) divergent

(c) Let N and N' be normed linear spaces and let T be a linear transformation of N into N'. Then

(i) T is bounded (ii) T is differentiable

(iii) T is discontinuous (iv) Non of these

(d) If N is a normed linear space and K is a convex subset of N, then $\overline{K}$

(i) is not a convex set
(ii) is a finite set
(iii) is a convex set
(iv) None of these.

(e) $(\alpha x + \beta y, z)$ is equal to

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(i) $\alpha(x,z) + \beta(y,z)$ (ii) $\bar{\alpha}(x,z) + \bar{\beta}(y,z)$
(iii) $\alpha(z,x) + \beta(z,y)$ (iv) $\bar{\alpha}(z,x) + \bar{\beta}(z,y)$.

(f) A closed convex subset C of a Hilbert space H consists a unique vector of

(i) largest Norm (ii) Smallest Norm
(iii) Smallest mode (iv) largest mode.

(g) The relation of orthogonality in a Hilbert space is

(i) reflexive (ii) transitive
(iii) anti-symmetric (iv) symmetric

(h) Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set, then condition is equivalent to one another is

(i) $\|x\|^2 = \sum |(x, e_i)|^2$, $x \in H$

(ii) $|x|^2 = \sum |(x, e_i)|^2$, $x \in H$

(iii) $x^2 = \sum |(x, e_i)|^2$, $x \in H$

(iv) $x^2 = \sum (x, e_i)^2$, $x \in H$.

(i) If H is a Hilbert space, Then H is
(i) symmetric (ii) reflexive
(iii) antisymmetri (iv) transitive.

(I) If L is an inner product space. Then $\sqrt{(x,x)}$

(i) Obeys arithematical properties
(ii) Obeys geometrical properties
(iii) Obeys harmonic properties
(iv) Obeys properties of norm.

PART B

Answer any four questions of the following $15 \times 4 = 60$

2 (a) Define Normed linear space and Prove that let N be a normed linear space and let d be a function from $N \times N$ into R defined by $d(x,y) = \|x-y\|$. Then d is a metric on N.

(b) let $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ be sequences of scalars such that $\underset{n\to\infty}{Lt}\, x_n = x_0$ and $\underset{n\to\infty}{Lt}\, y_n = y_0$, where x_0, y_0 are scalars and let α be any scalar, then Prove that

(i) $\underset{n\to\infty}{Lt} (x_n + y_n) = x_0 + y_0$

(ii) $\underset{n\to\infty}{Lt}\, \alpha x_n = \alpha x_0$

(iii) $\underset{n\to\infty}{Lt} (x_n - y_n) = x_0 - y_0$

3. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots y_n)$ denote n-tuples of scalars (real or complex). Define $\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$ for $p \geq 1$

Then Prove that

(i) $\sum_{i=1}^{n} |x_i y_i| \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} |y_i|^q\right]^{\frac{1}{q}}$

$= \|x\|_p \, \|y\|_q$

and (ii) If $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then Prove that

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p\right]^{\frac{1}{p}}$$

i.e $\|x+y\|_p \leq \|x\|_p + \|y\|_p$, where $1 \leq p < \infty$.

4. Let p be a real number such that $1 \leq p < \infty$ and let l_p denote the space of all sequences $x = \langle x_1, x_2, \ldots x_n \ldots \rangle$ of scalars such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$. Show that l_p is a Banach space under the norm $\|x\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{\frac{1}{p}}$

5. Define quotient space and Prove that, if M is a closed linear manifold (subspace) in a normed linear space N. For each coset $x+M$ in the quotient space $\frac{N}{M}$

We define $\|x+M\| = \inf\{\|x+M\| : x \in M\}$, then $\|x+M\|$ is a norm on $\frac{N}{M}$ and thus $\frac{N}{M}$ is a normed linear space.

6. Define Hilbert space and Prove that in a Hilbert space

(i) $(\alpha x - \beta y, z) = \alpha(x, z) - \beta(y, z)$

(ii) $(x, \beta y + \gamma z) = \bar{\beta}(x, y) + \bar{\gamma}(x, z)$

(iii) $(x, \beta y - \gamma z) = \bar{\beta}(x, y) - \bar{\gamma}(x, z)$

(iv) $(x, 0) = 0, \forall x \in H$ and $(0, x) = 0, \forall x \in H$.

7. If x and y are any two vectors in a Hilbert space, then Prove that

(i) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

and (ii) $4(x, y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

8 (a) Prove that a closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

(b) Let S be a non-empty subset of a Hilbert space H, then $S^\perp$ is a closed linear subspace of H.

9. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H.

Then the following conditions are equivalent to one and other

(i) $\{e_i\}$ is complete

(ii) $x \perp \{e_i\} \Rightarrow x = 0$

(iii) If x is an arbitrary vector in H, then $x = \sum (x, e_i) e_i$

(iv) If x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$.

x

Answers of objectives

Q No. (1)

a $\longrightarrow$ (ii)
b $\longrightarrow$ (iii)
c $\longrightarrow$ (i)
d $\longrightarrow$ (iii)
e $\longrightarrow$ (i)
f $\longrightarrow$ (iii)
g $\longrightarrow$ (iv)
h $\longrightarrow$ (i)
i $\longrightarrow$ (ii)
j $\longrightarrow$ (iii)

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SET – TWO

MODAL QUESTIONS OF

P.G. Sem-III, Paper – 308

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CHAIBASA

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The figures in the margin indicate full marks.

Answer from both the parts A and B. Part A contains multi choice questions.

PART A

1. Answer all questions $1 \times 10 = 10$

(a) $||\alpha x|| = |\alpha|\,||x||$ is called

(i) absolute homogenity (ii) sub additivity

(iii) Linearity (iv) None of these

(b) A normed linear space is called complete iff every Cauchy sequence in N is

(i) Oscillatory (ii) Convergent
(iii) divergent (iv) None of these

(c) L_p is a linear space under the operations
(i) $(f+g)x = f(x)+g(x)$, $(\alpha f)x = \alpha f(x)$
(ii) $(fg)x = f(x)+g(x)$, $(\alpha f)x = \alpha f(x)$
(iii) $|f+g|(x) = |f(x)|+|g(x)|$, $f(x) = g(x)$
(iv) $\|f+g\|(x) = \|f(x)\|+\|g(x)\|$, $f(x) = g(x)$
where $f, g \in L_p$, any scalar

(d) Every linear subspace of a linear space is
(i) Concave (ii) Complete
(iii) convex (iv) non Complete.

(e) Let $f, g \in L_p$, where $1 \leq p < \infty$. Then
(i) $\|f+g\|_p = \|f\|_p + \|g\|_p$
(ii) $\|f+g\|_p \geq \|f\|_p + \|g\|_p$
(iii) $\|f+g\|_p = \|f\|_p \cdot \|g\|_p$
(iv) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

(f) Let H be a complex Banach space. If a complex no. (x,y) is called inner product of x and y then

(i) $(x,x) = |x|^2$ (ii) $(x,x) = ||x||^2$

(iii) $(x,x) = x^2$ (iv) $(x,x) = x \cdot x$

(g) If S is a convex subset of a linear space L and α is any real number such that $0 \leq \alpha \leq 1$. Then

(i) $\alpha x + \alpha y \in S$ where $x, y \in S$

(ii) $(1+\alpha)x + \alpha y \in S$, where $x, y \in S$

(iii) $\alpha x - \alpha y \in S$, where $x, y \in S$

(iv) $(1-\alpha)x + \alpha y \in S$, where $x, y \in S$

(h) If x and y are two vectors in a Hilbert space, then

(i) $||x+y||^2 - ||x-y||^2 = 4 Re(x,y)$

(ii) $||x+y||^2 + ||x-y||^2 = 4 Re(x,y)$

(iii) $||x+y||^2 - ||x-y||^2 = 4 Im(x,y)$

(iv) $||x+y||^2 + ||x-y||^2 = 4 Im(x,y)$

(i) If x is orthogonal to y, then every scalar multiple of x

(i) is equal to y (ii) is greater than y

(iii) is orthogonal to y (iv) is less than y.

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(j) If M is a linear subspace of a Hilbert space H, The M is closed iff

(i) $M = M^{\perp}$ (ii) $M = M^{\perp\perp}$

(iii) $M = M^{\perp\perp\perp}$ (iv) $\overline{M} = M^{\perp}$

PART B

Answer any four questions of the following $15 \times 4 = 60$

2 (a) Let N be a normed linear space and let F denote C or R. Then the mappings

$f : N \times N \to N : f(x,y) = x+y$

and $g : F \times N \to N : g(\alpha, x) = \alpha x$, where $\alpha \in F$.

are continuous. 5

(b) Let N be a normed linear space and let $x, y \in N$. Then prove that

$\big| \|x\| - \|y\| \big| \leq \|x-y\|$ 5

(c) Let N be a normed linear space with the norm $\| \ \|$. Then the mapping

$f : N \to R : f(x) = \|x\|$ is continuous, that is the norm $\| \ \|$ on N is a continuous function 5

3. Let $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ be sequence of scalars such that

$\sum_{n=1}^{\infty} |x_n|^p < \infty$ and $\sum_{n=1}^{\infty} |y_n|^p < \infty$ for $p \geq 1$

Define $\|x\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{\frac{1}{p}}$. Then Prove that

(i) $\sum_{n=1}^{\infty} |x_n y_n| \leq \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{n=1}^{\infty} |y_n|^p\right]^{\frac{1}{q}}$

$$= \|x\|_p \|y\|_q$$

Where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

(ii) $\left[\sum_{n=1}^{\infty} | x_n + y_n |^p\right]^{\frac{1}{p}} \leq \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{\frac{1}{p}} + \left[\sum_{n=1}^{\infty} |y_n|^p\right]^{\frac{1}{p}}$

ie $\|x+y\|_p \leq \|x\|_p + \|y\|_p$, where $1 \leq p < \infty$

4. State and Prove Riesz - Fisher Theorem

5. Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. Then f can be extended to a functional F defined on the whole space N such that $\|F\| = \|f\|$.

6. (a) If x and y are any two vector in a Hilbert space H then Prove that
$$|(x, y)| \leq \|x\| \cdot \|y\|.$$ 5

(b) In a Hilbert space, the inner product is jointly continuous. Prove it. 5

(c) If L is a inner product space, show that $\sqrt{(x,x)}$ has the properties of a norm. 5.

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7. (a) If S, S_1, S_2 are non empty subsets of a Hilbert space H, then prove the following

(i) $\{0\}^{\perp} = H$ (ii) $H^{\perp} = \{0\}$

(iii) $S \cap S^{\perp} \subset \{0\}$ (iv) $S_1 \subset S_2 \Rightarrow S_2^{\perp} \subset S_1^{\perp}$

(v) $S \subset S^{\perp\perp}$

(b) If M is a linear subspace of a Hilbert space, Then show that M is closed $\Leftrightarrow M = M^{\perp\perp}$.

8. Show that the mapping $\phi : H \to H^*$ defined by $\psi(y) = f_y$, where $f_y(x) = (x, y) \; \forall x \in H$ is one-to-one, onto, additive but not linear, and an isometry.

9. Let $\{e_1, e_2, \cdots, e_n\}$ be a finite orthonormal set in Hilbert space H. If x is any vector in H, then prove that

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$$

X

Answer of objective Q 1

a — (i)	f — (ii)
b — (ii)	g — (iv)
c — (i)	h — (i)
d — (iii)	i — (iii)
e — (iv)	J — (ii)

SET Three

MODAL QUESTIONS
OF
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Answer from both the parts A and B.

Part A contains multiple choice questions.

PART – A

1. Answer all questions $1 \times 10 = 10$

(a) If the condition $\|x\| = 0$ is dropped from the definition of normed linear space N, then N is called

(i) Subspace (ii) Banach space

(iii) Pseudo-norm (iv) Symmetric norm

(b) In a normed linear space, every convergent sequence is a

(i) ~~bounded sequence~~

(ii) unbounded sequence

(iii) Cauchy sequence

(iv) Complete sequence

(c) A normed linear space which is complete as a metric space is called

(i) Banach space (ii) Hilbert space

(iii) Linear space (iv) Topological space

(d) Let $x = \langle x_n \rangle$ and $y = \langle y_n \rangle$ be sequences of scalars and α be any scalar. Then which is not true?

(i) $x + y = \langle x_n + y_n \rangle$

(ii) $xy = \langle x_n \cdot y_n \rangle$

(iii) $\alpha x = \langle \alpha x_n \rangle$

(iv) $\alpha - x = \langle \alpha - x_n \rangle$.

(e) Let $f, g \in L_p$, where $1 \leq p < \infty$. Then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ is called

(i) Cauchy's inequality

(ii) Minkowski's inequality

(iii) Schwartz's inequality

(iv) Holder's inequality

(f) The L_p Spaces are

(i) Linear space (ii) Normed linear space

(iii) Hilbert space (iv) Banach space.

(g) In a Hilbert space H, which is true?

(i) $(x,0) = 0, \forall x \in H$ and $(0,x) = 0, \forall x \in H$.

(ii) $(x,0) = 1, \forall x \in H$ and $(0,x) = 1, \forall x \in H$

(iii) $(x,0) = -1, \forall x \in H$ and $(0,x) = -1, \forall x \in H$

(iv) $(x,0) = \infty, \forall x \in H$ and $(0,x) = \infty, \forall x \in H$

(h) If x and y are any two vectors in a Hilbert space, then which is true?

(i) $(x,y) = I_m(x,y) + i Re(x,y)$

(ii) $(x,y) = Re(x,y) + I_m(x,y)$

(iii) $(x,y) = Re(x,y) + I_m(x,iy)$

(iv) $(x,y) = Re(x,y) + i Re(x,iy)$

(i) The zero vector is only vector which is
(i) ~~complement to 1~~
(ii) orthonormal to itself
(iii) orthogonal to itself
(iv) None of these

(J) If M is a closed linear subspace of a Hilbert space H. Then

(i) $H = M \oplus M^{\perp}$ (ii) $H = M \oplus M^{\perp\perp}$

(iii) $H = M \oplus \overline{M}$ (iv) $H = M \oplus M^{\perp\perp\perp}$

PART B

Answer any four questions of the following $15 \times 4 = 60$

2. Let p be a real number such that $1 \leq p < \infty$ and denote by l_p^n the space of all n-tuples $x = (x_1, x_2, x_3, \cdots, x_n)$ of scalars. Show that l_p^n is a Banach space under the norm $\|x\|^p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$.

3. In the linear space $C[0,1]$ of real valued continuous functions on $[0,1]$, define two norms as follows:

$$\|f\|_1 = \max_{0 \leq t \leq 1} |f(t)|,$$

$$\|f\|_2 = \int_0^1 |f(t)|\, dt$$

Prove that these are actually norms, that the first makes the linear space into a Banach space and second does not.

4 (a) Let B and B' be Banach spaces. If T is a continuous linear transformation of B onto B', then T is an open mapping.

(b) Let B and B' be Banach spaces and let T be a linear transformation of B into B', Then prove that T is a continuous mapping if and only if its graph is closed.

5. If B is a complex Banach space whose norm obeys the parallelogram law and if an inner product is defined on B by $4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$ then B is a Hilbert space.

6 (a) Let M be a closed linear subspace of a Hilbert space H. Let x be a vector not in M and let d be the distance from x to M. Then there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

(6) If M is a linear subspace of a Hilbert space H, Show that M is closed $\iff M = M^{\perp\perp}$

(7) If $\{e_i\}$ is an orthonormal set in a Hilbert space H, and if x is an arbitrary vector in H, then prove that

$$x - \sum (x, e_i) e_i \perp e_j \text{, for each } j$$

(8) Prove that, if H is a Hilbert space, then H is reflexive.

(9) State and Prove Gram-Schmidt orthogonalization process.

— X —

Answers of objectives Q No. 1

a — (iii)
b — (iii)
c — (i)
d — (iv)
e — (ii)
f — (ii)
g — (i)
h — (iv)
i — (iii)
J — (i)

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